The nonisentropic generalisation of the classical theory of Riemann invariants

1987 J. Phys. A: Math. Gen. 20 2721

(http://iopscience.iop.org/0305-4470/20/10/016)

J. Phys. A: Math. Gen. 20 (1987) 2721-2731. Printed in the UK

The non-isentropic generalisation of the classical theory of Riemann invariants

B Gaffet

CNRS and CEN Saclay, Service d'Astrophysique, 91191 Gif-sur-Yvette Cedex, France

Received 8 August 1986

Abstract. The one-dimensional motion of an isentropic gas has long been known to be characterised by the presence of Riemann invariants; the choice of the latter as independent variables results in an essential simplification of the system of Euler equations which becomes linear. Another case which also presents Riemann invariants was discovered about 1954 by Stanyukovich and by Martin and Ludford; the LMS gas and the isentropic gas have been shown to be mathematically equivalent. These two celebrated cases of integrability are shown here to be but particular cases of a more general class of entropy distributions, for which Riemann invariants are here constructed.

1. Introduction

The compressible flow of an ideal gas is governed by the well known Euler equations, which constitute a hyperbolic system possessing two distinct systems of characteristic curves, (C^+) and (C^-). These curves play an essential role both from the point of view of the physics and of the mathematical analysis and it is usually very convenient to choose characteristic coordinates (α, β) as independent variables, rather than the Eulerian coordinates, r, t, or the Lagrangian coordinates M, t (M here denotes the Lagrangian mass). The (C^+) curves are then those on which α remains constant and the (C^-) curves those where β is constant. From that definition, it is apparent that α, β are only defined up to a transformation of the general form

$$\alpha' = f(\alpha) \qquad \beta' = g(\beta) \tag{1.1}$$

where the functions f and g are both arbitrary. Thus the system of Euler equations, which is of second order when the entropy distribution is specified, becomes a fourth-order system in the characteristic formalism, owing to the presence of the two arbitrary functions. There exists however a remarkable exception, which is the case where a special choice of characteristic coordinates I^+, I^- may be selected out of the whole set (1.1), i.e. when $I^\pm$ may be determined explicitly and unambiguously. These may be called Riemann invariants, after the work of Riemann [1]. In such a case the characteristic equations written with respect to the variables $I^\pm$ remain second order.

Another advantage of the presence of Riemann invariants (RI) is that, at least in certain favourable cases, they directly lead to an expression of the general integral in closed form, as pointed out by Gaffet [2], namely when the Euler equations present a symmetry, denoted (T) (i.e. (T) is a contact transformation or, more generally, a Bäcklund transformation) the RI $I^\pm$ transform into (other) quantities $K^\pm$, but the characteristic curves themselves remain unchanged [3]†. Under such conditions the quantities $I^{\pm}$, $K^{\pm}$ must be related by an equation of the form of (1.1):

$$K^{\pm}=f^{\pm}(I^{\pm}). \tag{1.2}$$

If the quantities $K^{\pm}$ and $I^{\mp}$ are independent, as is frequently the case, the $f^{\pm}$ cannot be fixed functions and therefore are 'arbitrary', i.e. their choice depends upon the initial and boundary data. Then (1.2) constitutes an expression of the general integral in closed form ([2], equation (5.3)).

2. The two cases already known to possess Riemann invariants

Following the notation of [2], we write the Euler equations for a polytrope of index γ in the following form:

$$\begin{aligned} &P=\rho^{\gamma}\sigma(M) \\ &\mathrm{d}M=\rho(\mathrm{d}r-v\,\mathrm{d}t) \\ &v_t+P_M=0 \end{aligned} \tag{2.1}$$

where P, ρ, v denote the pressure, density and fluid velocity, M is the Lagrangian mass, and subscripts indicate partial differentiation; σ may be called the adiabatic invariant. In cases where the distribution $\sigma(M)$ is power law we define the 'entropy index' b as

$$\sigma(M)\propto 1/M^{b}. \tag{2.2}$$

The sound velocity is related to P and ρ by the well known formula

$$c^2=\gamma P/\rho. \tag{2.3}$$

The characteristic formulation may be derived by the method of Courant and Friedrichs [4], and has already been given in [2]; we shall repeat it here for convenience:

$$r_\alpha=(v-c)t_\alpha \tag{2.4a}$$

$$P_\alpha=+\rho c v_\alpha \tag{2.4b}$$

$$M_\alpha=-\rho c t_\alpha \tag{2.4c}$$

plus three corresponding equations with respect to the variable β, obtained by changing c to $-c$. It is of interest to note another form equivalent to (2.4b):

$$(\gamma-1)v_\alpha-2c_\alpha=-b(P/M)t_\alpha \tag{2.5}$$

valid for a power law entropy distribution.

2.1. *The case of isentropic flow [1]*

The derivation of Riemann invariants becomes straightforward if one starts from the characteristic formulation (2.4). In the isentropic case $b=0$ and equation (2.5) may be integrated to give

$$v-2c/(\gamma-1)=g(\beta).$$

† This result has an obvious physical interpretation: two solutions which differ only slightly, and only in the neighbourhood of one characteristic, must under the transformation (T) go over into two new solutions which also coincide almost everywhere, i.e. everywhere except near one characteristic. That is, a characteristic must be transformed into another characteristic.

Hence the two RI:

$$I^{\pm} = v \pm 2c/(\gamma - 1). \tag{2.6}$$

It is easily verified that eliminating M from equation (2.4c) yields an equation for $t(I^+, I^-)$ which is not only of second order but also linear.

The cases of isentropic flow where $\gamma = (2n+1)/(2n-1)$, with n integer, are known to be mathematically equivalent to the case $\gamma = 3$ [5].

For the latter case ($\gamma = 3$) we have shown in [2] that there exists an exact symmetry (T^*) of the Euler equations having the properties $c^* = ct$, $v^* = vt - r$. Thus (T^*) transforms $I^{\pm} \equiv (v \pm c)$ into a new pair of Riemann invariants $K^{\pm} \equiv (v \pm c)t - r$ and as a result the general solution may be written down in closed form in the following way:

$$(v \pm c)t - r = f^{\pm}(v \pm c). \tag{2.7}$$

Equation (2.7) is a classical result which may also be found in [5].

2.2. *The Ludford-Martin-Stanyukovich* (LMS) *gas*

This is the case characterised by a power law entropy profile with index

$$b = 3\gamma - 1. \tag{2.8}$$

It was first considered by Stanyukovich [6], who showed that it is mathematically equivalent to the isentropic case ($b = 0$). Later on, this problem was more extensively studied [7, 8 p 200, 9, 10]. In particular, Martin and Ludford [7] have noted the presence of a pair of Riemann invariants, which are

$$I^{\pm} = M\left(v \pm \frac{2c}{(\gamma - 1)}\right) - \Pi \tag{2.9}$$

where Π represents the first integral of momentum, defined up to an arbitrary additive constant by

$$\mathrm{d}\Pi = v\,\mathrm{d}M - P\,\mathrm{d}t \tag{2.10}$$

or, in characteristic form ([2], equation (3.13))

$$\Pi_\alpha = (v + c/\gamma)M_\alpha. \tag{2.11}$$

The transformation, which we denoted by ($\bar{T}$), relating the isentropic and LMS gases, can be interpreted in terms of an M-dependent rescaling of the system of units, as shown in [2].

3. New cases of one-dimensional gas flow possessing Riemann invariants

Our starting point is that generalised Riemann invariants may also be constructed for the case

$$P \propto \rho^3/M^4 \tag{3.1}$$

i.e. when $\gamma = 3$ and $b = 4$. However, it has also been shown by Gaffet [11] that such an equation of state is mathematically equivalent to the more general entropy distribution

$$P = \rho^3(a_0 + a_1 M + a_2 M^2)^{-4} \tag{3.2}$$

where a_0, a_1 and a_2 are arbitrary constants. Therefore, RI will also be present for the three-parameter class of entropy profiles (3.2). We now proceed with the derivation, assuming the power law profile (3.1).

3.1. *The characteristic coordinates $\hat{\alpha}$, $\hat{\beta}$*

It is worth recalling here that the quantity

$$v^* \equiv vt - r \tag{3.3}$$

plays a role completely analogous to that of the velocity v itself [2, 12]. Then the first step in our derivation of the Riemann invariants will be to show that the expressions

$$\begin{aligned} \hat{\alpha} &\equiv M(v_\alpha v^*_{\alpha\alpha} - v^*_\alpha v_{\alpha\alpha}) \\ \hat{\beta} &\equiv M(v_\beta v^*_{\beta\beta} - v^*_\beta v_{\beta\beta}) \end{aligned} \tag{3.4}$$

are characteristic coordinates too.

In view of the complete symmetry of the roles played by (C^+) and (C^-) characteristics, it will be sufficient to give the proof for $\hat{\beta}$ say, i.e. we must calculate

$$\hat{\beta}_\alpha \equiv M_\alpha[v_\beta v^*_{\beta\beta} - v^*_\beta v_{\beta\beta}] + M[v_\beta v^*_{\beta\beta} - v^*_\beta v_{\beta\beta}]_\alpha \tag{3.5}$$

and show that it vanishes identically.

The above expression involves second-order derivatives, both of the mixed type, such as $v_{\alpha\beta}$, $v^*_{\alpha\beta}$, and also the unmixed derivatives $v_{\beta\beta}$, $v^*_{\beta\beta}$. It also involves third-order derivatives of the type $v_{\alpha\beta\beta}$, $v^*_{\alpha\beta\beta}$. We show in the appendix that the second-order derivatives of mixed type can all be expressed in terms of the first-order derivatives only and as a consequence the third-order derivatives involved can all be reduced to second-order unmixed derivatives such as $v_{\beta\beta}$, $v^*_{\beta\beta}$. We also show in the appendix that the coefficients of the latter in (3.5) are identically zero and that the remaining expression involving first-order derivatives vanishes identically as well. This completes the proof that $\hat{\beta}$ is a function of β only, and hence a characteristic coordinate. One can show in the same way that $\hat{\alpha}$ is a characteristic coordinate as well.

3.2. *The pair of Riemann invariants $I^\pm$*

In view of the fact that $\hat{\beta}$, as defined by (3.4), is cubic in the differential $d\beta$, it is natural to form the quantity

$$\hat{\beta}^{1/3} = g(\beta) \tag{3.6}$$

and its primitive

$$I^- = \int g(\beta)\, d\beta. \tag{3.7}$$

Clearly, I^- is a characteristic coordinate as well and furthermore it is independent, by construction, of the choice of β. In fact, the only ambiguity remaining in the definition of I^- is the presence of an arbitrary additive constant, the integration constant. Therefore I^- is a Riemann invariant.

It is possible to rewrite it in a manifestly intrinsic form by returning to the Lagrangian independent variables M, t. The transformation of variable (from α, β to M, t) may be performed by means of the following relations ([2], equation (3.12)):

$$\begin{aligned}\partial_\alpha &= t_\alpha(\partial_t - \rho c \partial_M) \\ \partial_\beta &= t_\beta(\partial_t + \rho c \partial_M)\end{aligned} \tag{3.8}$$

where the symbol ∂ denotes the operator of partial differentiation and

$$\begin{aligned}2t_\alpha\, \mathrm{d}\alpha &= \mathrm{d}t - \frac{1}{\rho c}\,\mathrm{d}M \\ 2t_\beta\, \mathrm{d}\beta &= \mathrm{d}t + \frac{1}{\rho c}\,\mathrm{d}M.\end{aligned} \tag{3.9}$$

It is useful to introduce a quantity h^- whose definition is intrinsic:

$$h^- \equiv \frac{v^*_\beta}{v_\beta} = \frac{v^*_t + \rho c v^*_M}{v_t + \rho c v_M}. \tag{3.10}$$

In terms of h^-, we have

$$v_\beta v^*_{\beta\beta} - v^*_\beta v_{\beta\beta} \equiv v^2_\beta h^-_\beta \tag{3.11}$$

and hence

$$\begin{aligned}\mathrm{d}I^-/\mathrm{d}\beta &= M^{1/3} v_\beta^{2/3} (h^-_\beta)^{1/3} \\ &= M^{1/3} t_\beta (v_t + \rho c v_M)^{2/3} (h^-_t + \rho c h^-_M)^{1/3}.\end{aligned} \tag{3.12}$$

Furthermore, $\partial I^-/\partial\alpha = 0$ as has already been shown.

Thus we obtain the following expression for I^-, which is manifestly intrinsic:

$$\mathrm{d}I^- = \tfrac{1}{2} M^{1/3} (v_t + \rho c v_M)^{2/3} (h^-_t + \rho c h^-_M)^{1/3} (\mathrm{d}t + (1/\rho c)\,\mathrm{d}M). \tag{3.13}$$

A corresponding expression may be derived for I^+ by changing c to $-c$ everywhere (including in the definition of h^-, which becomes h^+).

It is worth noting that I^- is the potential associated with a conservation law, in the same way that Π in (2.10) is the potential describing momentum conservation ([2], § 3.2). The conserved current flows along the equipotentials, which in the present case are the characteristics.

In conclusion we have obtained two new conservation laws, together with the two potentials I^+, I^-.

4. The case of arbitrary entropy distributions

We restrict ourselves here to the consideration of polytropes $\gamma = 3$.

4.1. A new Lagrangian variable

For arbitrary entropy distributions, it will be convenient to rewrite the system of characteristic equations (2.4) in the form given in [2] (equations (3.2)-(3.5)):

$$r_\alpha = (v - c)t_\alpha \qquad r_\beta = (v + c)t_\beta \tag{4.1a}$$

$$(v_\alpha - c_\alpha) = +\mu c^3 t_\alpha \qquad (v_\beta + c_\beta) = +\mu c^3 t_\beta \tag{4.1b}$$

$$\Psi_\alpha = -c^2 t_\alpha \qquad \Psi_\beta = +c^2 t_\beta \tag{4.1c}$$

where

$$d\Psi = \sqrt{3\sigma}\, dM \tag{4.2}$$

and μ is a Lagrangian variable related to the slope of the entropy profile:

$$\mu = \frac{1}{6}\frac{d\ln\sigma}{d\Psi}. \tag{4.3}$$

We note the following identity relating first-order derivatives:

$$(v_\beta + c_\beta)t_\alpha = (v_\alpha - c_\alpha)t_\beta. \tag{4.4}$$

Now, as a generalisation of the expression (3.4) we propose for arbitrary $\sigma(M)$:

$$\begin{aligned} B &\equiv \sigma(M)^{-1/4}(v_\beta^* v_{\beta\beta} - v_\beta v_{\beta\beta}^*) \\ &\equiv \sigma^{-1/4}(H_1 + cH_2) \end{aligned} \tag{4.5}$$

with

$$\begin{aligned} H_1 &\equiv v_\beta t_\beta (c_\beta - v_\beta) \\ H_2 &\equiv (v_\beta t_{\beta\beta} - t_\beta v_{\beta\beta}). \end{aligned} \tag{4.6}$$

In the same way as in § 3 (see also the appendix) we now calculate the α derivative of B. We have

$$\sigma^{1/4}B_\alpha = H_{1\alpha} + \frac{3H_1}{2c}(v_\alpha - c_\alpha) + \sigma^{1/4}\{\sigma^{-1/4}cH_2\}_\alpha \tag{4.7}$$

and, of course, B_α will not in general vanish.

As in § 3, the second-order mixed derivatives appearing in (4.7) can be reduced to first-order derivatives, through the following reduction formulae:

$$\begin{aligned} ct_{\alpha\beta} &= -(c_\alpha t_\beta + c_\beta t_\alpha) \\ &= +(v_\beta t_\alpha - v_\alpha t_\beta) \end{aligned} \tag{4.8}$$

$$2cv_{\alpha\beta} = (v_\alpha c_\beta + v_\beta c_\alpha). \tag{4.9}$$

We note that $v_{\alpha\beta}$ and $t_{\alpha\beta}$ are 'proportional':

$$2v_{\alpha\beta} = -\mu c^3 t_{\alpha\beta}. \tag{4.10}$$

For $c_{\alpha\beta}$ we find

$$\begin{aligned} \tfrac{2}{3}cc_{\alpha\beta} &= (c_\alpha v_\beta - v_\alpha c_\beta + 2c_\alpha c_\beta) + \nu(v_\alpha - c_\alpha)(v_\beta + c_\beta) \\ &= (v_\alpha c_\beta - v_\beta c_\alpha + 2v_\alpha v_\beta) + (\nu - 2)(v_\alpha - c_\alpha)(v_\beta + c_\beta) \end{aligned} \tag{4.11}$$

where the new Lagrangian coordinate ν is related to the second derivative of $\sigma(M)$ as

$$\nu \equiv \frac{2}{3}\frac{d(1/\mu)}{d\Psi}. \tag{4.12}$$

The following relation also turns out to be useful:

$$2c(v_{\alpha\beta} - \tfrac{1}{3}c_{\alpha\beta}) = (v_\alpha - c_\alpha)[2c_\beta - \nu(v_\beta + c_\beta)]. \tag{4.13}$$

Concerning the double β derivatives, only two of them are found to be independent. For example, the derivatives $v_{\beta\beta}$, $c_{\beta\beta}$, $t_{\beta\beta}$ are linked by the relation

$$t_\alpha(v_{\beta\beta}+c_{\beta\beta})+(c_\alpha-v_\alpha)t_{\beta\beta}=t_\beta(3v_{\alpha\beta}-c_{\alpha\beta}) \tag{4.14}$$

valid independent of the entropy profile σ.

Now third-order derivatives arise from the last term in (4.7) only:

$$\begin{aligned}2\sigma^{1/4}(\sigma^{-1/4}cH_2)_\alpha &= 2c(v_\beta t_{\alpha\beta\beta}-t_\beta v_{\alpha\beta\beta})\\ &\quad +2c(v_{\alpha\beta}t_{\beta\beta}-t_{\alpha\beta}v_{\beta\beta})+(3v_\alpha-c_\alpha)(v_\beta t_{\beta\beta}-t_\beta v_{\beta\beta}).\end{aligned} \tag{4.15}$$

The third-order term can be changed to second order by means of the reduction formulae (4.8)-(4.11):

$$\begin{aligned}&2c(v_\beta t_{\alpha\beta\beta}-t_\beta v_{\alpha\beta\beta})\\ &\quad =(3v_\beta+c_\beta)(t_\alpha v_{\beta\beta}-v_\alpha t_{\beta\beta})+2v_\beta(v_\beta-c_\beta)t_{\alpha\beta}+(t_\beta/t_\alpha)c_{\alpha\beta}(v_\alpha t_\beta-v_\beta t_\alpha)\\ &\quad\quad +(t_\beta/t_\alpha)v_{\alpha\beta}[(c_\beta-2v_\beta)t_\alpha-3v_\alpha t_\beta].\end{aligned} \tag{4.16}$$

The second term in (4.15) is similarly reduced to

$$2c(v_{\alpha\beta}t_{\beta\beta}-t_{\alpha\beta}v_{\beta\beta})=(3v_\beta+c_\beta)(v_\alpha t_{\beta\beta}-t_\alpha v_{\beta\beta})+(c_\alpha-3v_\alpha)(v_\beta t_{\beta\beta}-t_\beta v_{\beta\beta}). \tag{4.17}$$

Thus it is found that all second-order β-derivative terms cancel out in the expression (4.15), which is therefore expressible in terms of first-order derivatives only:

$$\begin{aligned}2\sigma^{1/4}(\sigma^{-1/4}cH_2)_\alpha &\equiv 2v_\beta(v_\beta-c_\beta)t_{\alpha\beta}+(t_\beta/t_\alpha)c_{\alpha\beta}(v_\alpha t_\beta-v_\beta t_\alpha)\\ &\quad +(t_\beta/t_\alpha)v_{\alpha\beta}[(c_\beta-2v_\beta)t_\alpha-3v_\alpha t_\beta].\end{aligned} \tag{4.18}$$

The evaluation of the remaining terms in (4.7) is straightforward. We find

$$H_{1\alpha}=t_\beta(c_\beta-2v_\beta)v_{\alpha\beta}+v_\beta t_\beta c_{\alpha\beta}+v_\beta(c_\beta-v_\beta)t_{\alpha\beta} \tag{4.19}$$

and hence the result

$$\begin{aligned}2\sigma^{1/4}B_\alpha &\equiv (t_\beta/t_\alpha)c_{\alpha\beta}(v_\alpha t_\beta+v_\beta t_\alpha)+(3t_\beta/t_\alpha)v_{\alpha\beta}[t_\alpha(c_\beta-2v_\beta)-v_\alpha t_\beta]\\ &\quad +(3/c)v_\beta t_\beta(v_\alpha-c_\alpha)(c_\beta-v_\beta).\end{aligned} \tag{4.20}$$

(We note that the $t_{\alpha\beta}$ terms have cancelled out in the above equation.) After reduction of the second-order derivatives $v_{\alpha\beta}$, $c_{\alpha\beta}$, we obtain the following rather simple result:

$$B_\alpha=\frac{3(\nu-1)}{4c\sigma^{1/4}}t_\beta(v_\beta+c_\beta)(2v_\alpha v_\beta+v_\alpha c_\beta-v_\beta c_\alpha). \tag{4.21}$$

In the particular case where $\nu\equiv 1$, which is the entropy profile $\sigma(M)\propto 1/M^4$, B_α identically vanishes and the essential result of § 3 is recovered. For arbitrary σ, however, B_α will not in general vanish; still, its consideration leads to an important new result, as we presently show.

We introduce a new quantity:

$$\tilde{B}\equiv +(c^6/\sigma)t_\beta^3\equiv +\Psi_\beta^3/\sigma. \tag{4.22}$$

Its α derivative is easily obtained as

$$\tilde{B}_\alpha=\frac{+3}{c\sigma\mu^2}t_\beta(v_\beta+c_\beta)(2v_\alpha v_\beta+v_\alpha c_\beta-v_\beta c_\alpha). \tag{4.23}$$

We observe that B_α and $\tilde{B}_\alpha$ are proportional:

$$B_\alpha/\tilde{B}_\alpha = \tfrac{1}{4}\sigma^{3/4}\mu^2(\nu-1) \tag{4.24}$$

and the proportionality coefficient clearly is a Lagrangian variable, as are σ, μ and ν.

In conclusion, the ratio $B_\alpha/\tilde{B}_\alpha$ is a Lagrangian variable for arbitrary entropy distributions.

4.2. *The three-parameter class leading to Riemann invariants*

In the particular case where the new Lagrangian variable is a constant

$$4B_\alpha/\tilde{B}_\alpha = k \tag{4.25}$$

we obtain as new characteristic coordinates:

$$\hat{\beta} = (4B - k\tilde{B}) \tag{4.26}$$

from which a Riemann invariant, I^-, can be derived in the manner already illustrated in § 3:

$$I^- = \int \hat{\beta}^{1/3}\,\mathrm{d}\beta \tag{4.27}$$

and of course another Riemann invariant I^+ may also be found owing to the symmetry of the roles played by the two variables α and β.

The condition for the existence of that pair of RI is that a second order ordinary differential equation (ODE) be satisfied by the entropy profile σ:

$$\sigma^{3/4}\mu^2(\nu-1) = k. \tag{4.28}$$

We recall that μ and ν are defined by (4.2), (4.3) and (4.12). Equation (4.28) is solved by taking

$$G \equiv (3\sigma)^{-1/4} \tag{4.29}$$

as the unknown function and the best choice is to retain the Lagrangian mass coordinate M as independent variable. In this way we find

$$\begin{aligned} \mu &\equiv -\tfrac{2}{3}GG'(M) \\ \nu &\equiv (GG'' + G'^2)/G'^2 \end{aligned} \tag{4.30}$$

and hence

$$(3\sigma)^{3/4}\mu^2(\nu-1) \equiv \tfrac{4}{9}G''(M). \tag{4.31}$$

Thus the ODE (4.28) reduces to

$$G'' = \text{constant} \tag{4.32}$$

and is at once integrable in the form

$$G = (a_0 + a_1 M + a_2 M^2) \qquad \sigma \propto 1/G^4 \tag{4.33}$$

a_0, a_1 and a_2 being arbitrary constants.

For these entropy distributions, the Euler equations present a pair of Riemann invariants which are given by (4.27) and by another symmetrical equation.

5. Conclusion

The Euler equations describing one-dimensional adiabatic gas flow constitute a second-order system but generally become of fourth order† when characteristic coordinates are used instead of the Eulerian coordinates (r, t). There is an exception, however, which occurs when Riemann invariants exist, in which case the characteristic formulation remains second order. The original system may then be thought of as being reduced to its simplest form, being both second order and characteristic.

There exist only two types of gas for which Riemann invariants have been known to exist, up to now: the isentropic gas [1] and the so-called LMS gas [6, 7] which can be reduced to the former by means of a general transformation called $(\bar{T})$ [2].

We present a new three-parameter entropy distribution for which Riemann invariants are shown here to exist, assuming a polytrope $\gamma = 3$:

$$P/\rho^3 = (a_0 + a_1 M + a_2 M^2)^{-4}. \tag{5.1}$$

We have shown in an earlier work [11] that the above entropy distribution can be reduced by means of the transformation $(\bar{T})$ to the simple power law profile

$$P \propto \rho^3 / M^4. \tag{5.2}$$

Riemann invariants are explicitly constructed here, assuming the above entropy profiles.

Appendix. Proof that $\hat{\beta}$ is a characteristic coordinate

The aim of this appendix is to show that $\hat{\beta}$, defined by (3.4), is a characteristic coordinate, i.e. that the right-hand side of (3.5) vanishes identically. Substituting $v^* \equiv vt - r$, we write $\hat{\beta} \equiv MH_1 + McH_2$ with

$$\begin{aligned} H_1 &\equiv v_\beta t_\beta (c_\beta - v_\beta) \\ H_2 &\equiv (v_\beta t_{\beta\beta} - t_\beta v_{\beta\beta}). \end{aligned} \tag{A1}$$

Then the condition $\hat{\beta}_\alpha = 0$ is

$$M_\alpha H_1 + MH_{1\alpha} + (Mc)_\alpha H_2 + McH_{2\alpha} = 0. \tag{A2}$$

The term $H_{1\alpha}$ may be expanded as follows:

$$H_{1\alpha} = t_{\alpha\beta} v_\beta (c_\beta - v_\beta) + v_{\alpha\beta} t_\beta (c_\beta - 2v_\beta) + c_{\alpha\beta} t_\beta v_\beta. \tag{A3}$$

Note that it involves second-order mixed derivatives.

Starting from the characteristic equations (2.4) and (2.5), with $\gamma = 3$, $b = 4$, it is straightforward to establish the following reduction formulae for the mixed derivatives:

$$\begin{aligned} ct_{\alpha\beta} &= v_\beta t_\alpha - v_\alpha t_\beta \\ 2cv_{\alpha\beta} &= v_\alpha c_\beta + v_\beta c_\alpha \\ 2cc_{\alpha\beta} &= 3(v_\alpha v_\beta + c_\alpha c_\beta). \end{aligned} \tag{A4}$$

† The Euler equation and the continuity equation form a second-order system when Lagrangian coordinates M, t are used and the energy equation reduces to an algebraic relation between P, ρ and M. On the other hand, the characteristic system is of order four, since there are four independent characteristic equations of first order, namely (2.4b, c) plus the two corresponding equations with respect to β.

It is also important to notice the following relations between first-order derivatives:

$$(v_\beta + c_\beta)t_\alpha = (v_\alpha - c_\alpha)t_\beta \tag{A5a}$$

$$2cM_\alpha = 3M(v_\alpha - c_\alpha) \tag{A5b}$$

$$2(Mc)_\alpha = M(3v_\alpha - c_\alpha). \tag{A5c}$$

Equation (A5a) can easily be derived from (2.5), supplemented by the corresponding equation with respect to β.

Concerning the unmixed derivatives, it is found that only two of them are independent, i.e. we have relations of the type

$$\frac{(v_{\beta\beta} + c_{\beta\beta})}{(v_\beta + c_\beta)} = \frac{t_{\beta\beta}}{t_\beta} + \frac{3(c_\beta - v_\beta)}{2c}. \tag{A6}$$

Now, taking into account (A4), the second-order derivatives may be eliminated from the expression for $H_{1\alpha}$ with the result

$$2cH_{1\alpha} = 2t_\alpha v_\beta^2(c_\beta - v_\beta) + t_\beta[v_\alpha(5v_\beta^2 - 4v_\beta c_\beta + c_\beta^2) + 2c_\alpha v_\beta(2c_\beta - v_\beta)]$$

hence

$$\begin{aligned}(2c/M)(MH_1)_\alpha &\equiv 2c[H_{1\alpha} + (M_\alpha/M)H_1]\\ &= 2t_\alpha v_\beta^2(c_\beta - v_\beta) + t_\beta[v_\alpha(2v_\beta^2 - v_\beta c_\beta + c_\beta^2) + c_\alpha v_\beta(c_\beta + v_\beta)].\end{aligned} \tag{A7}$$

Subtracting from (A7) the quantity

$$v_\beta(c_\beta + v_\beta)[(c_\beta + v_\beta)t_\alpha + (c_\alpha - v_\alpha)t_\beta]$$

which, by (A5a), is identically zero, we derive the following simple and symmetrical result:

$$(2c/M)(MH_1)_\alpha = (3v_\beta^2 + c_\beta^2)(v_\alpha t_\beta - v_\beta t_\alpha). \tag{A8}$$

Next we calculate $H_{2\alpha}$, which involves the third-order derivatives $t_{\alpha\beta\beta}$, $v_{\alpha\beta\beta}$. Taking into account (A4), the latter may be reduced to the second-order derivatives $t_{\beta\beta}$, $v_{\beta\beta}$, as follows:

$$\begin{aligned} ct_{\alpha\beta\beta} &= (t_\alpha v_{\beta\beta} - v_\alpha t_{\beta\beta}) + \frac{(3v_\beta - c_\beta)}{2c}(v_\beta t_\alpha - v_\alpha t_\beta)\\ 2ct_\beta v_{\alpha\beta\beta} &= (v_\beta + c_\beta)\left[(v_\alpha t_{\beta\beta} - t_\alpha v_{\beta\beta}) + \frac{c_\beta}{c}(v_\alpha t_\beta - v_\beta t_\alpha)\right].\end{aligned} \tag{A9}$$

Next we obtain

$$v_\beta t_{\alpha\beta\beta} - t_\beta v_{\alpha\beta\beta} = \frac{(3v_\beta + c_\beta)}{2c}(t_\alpha v_{\beta\beta} - v_\alpha t_{\beta\beta}) + \frac{(3v_\beta^2 + c_\beta^2)}{2c^2}(v_\beta t_\alpha - v_\alpha t_\beta) \tag{A10}$$

from which $H_{2\alpha}$ may be computed:

$$\begin{aligned} H_{2\alpha} &\equiv (v_{\alpha\beta}t_{\beta\beta} - t_{\alpha\beta}v_{\beta\beta}) + (v_\beta t_{\alpha\beta\beta} - t_\beta v_{\alpha\beta\beta})\\ &= \frac{(3v_\alpha - c_\alpha)}{2c}(t_\beta v_{\beta\beta} - v_\beta t_{\beta\beta}) + \frac{(3v_\beta^2 + c_\beta^2)}{2c^2}(v_\beta t_\alpha - v_\alpha t_\beta).\end{aligned} \tag{A11}$$

In this way we obtain

$$\begin{aligned}(2c/M)(McH_2)_\alpha &\equiv (3v_\alpha - c_\alpha)cH_2 + 2c^2 H_{2\alpha}\\ &= (3v_\beta^2 + c_\beta^2)(v_\beta t_\alpha - v_\alpha t_\beta).\end{aligned} \tag{A12}$$

We note that even the second-order derivatives disappear altogether from the final result. Furthermore, comparing (A8) and (A12), we obtain the desired result (A2):

$$(MH_1)_\alpha + (McH_2)_\alpha = 0.$$

That proves that $\hat{\beta}$, as defined by (3.4), is a characteristic coordinate.

References

[1] Riemann B 1860 *Math. Phys. Klasse* **8** 43
[2] Gaffet B 1983 *J. Fluid Mech.* **134** 179
[3] Forsyth A R 1959 *Theory of Differential Equations* (New York: Dover)
[4] Courant R and Friedrichs K 1948 *Supersonic Flow and Shock Waves* (New York: Interscience)
[5] Landau L D and Lifshitz E M 1959 *Fluid Mechanics* (New York: Addison-Wesley)
[6] Stanyukovich K P 1954 *Dokl. Akad. Nauk.* **96** 441; 1960 *Unsteady Motion of Continuous Media* (Oxford: Pergamon)
[7] Martin M H and Ludford G S S 1954 *Commun. Pure Appl. Math.* **7** 45
[8] Cabannes H 1960 *Handbuch der Physik* vol IX (Berlin: Springer) pp 162–221
[9] Steketee J A 1972 *Q. J. Appl. Math.* **30** 167
[10] Steketee J A 1976 *J. Eng. Math.* **10** 69
[11] Gaffet B 1984 *Physica* **11D** 287
[12] Gaffet B 1981 *Preprint RIFP 442* Kyoto University